Final - Probability II (2024-25)
Time: 3 hours.
Attempt all questions. The total marks is 50.

1. The joint density X and Y is given by
$$f(x, y) = \frac{e^{-y}}{y}, \qquad 0 < x < y,\ 0 < y < \infty.$$
   (a) Compute $E[X^3|Y = y]$. **[4 marks]**
   (b) Compute $E[X^3]$. **[2 marks]**

2. Let X_1 and X_2 be independent exponential random variables, each having parameter λ.
   (a) Find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$. **[4 marks]**
   (b) Are Y_1 and Y_2 independent? **[1 mark]**

3. Consider a simple random walk S_n on $\mathbf{Z}$ starting at 0 with equal probability of going one unit to the right or to the left at each step. That is, $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$, where the X_i's are independent random variables taking the values $+1$ or -1 with probability $\frac{1}{2}$ each. What is the probability that the random walk does not hit -10 up to and including time 100 ? You can leave your answer as a finite sum. **[5 marks]**

4. Let $X_n, n \geq 1$ and X be random variables on a probability space. Say that X_n converges to X in L^p for some $p \geq 1$ if $E[|X_n - X|^p] \to 0$.
   (a) Show that $X_n \xrightarrow{L^p} X$ implies $X_n \xrightarrow{P} X$. **[3 marks]**
   (b) Give an example to show that $X_n \xrightarrow{L^p} X$ does not imply $X_n \xrightarrow{a.s.} X$. **[3 marks]**

5. Let X_1, X_2, X_3 be independent $U(0, 1)$ random variables. Let $Z = \max(X_1, X_2, X_3)$. Find $E(Z)$. **[5 marks]**

6. (a) State the Strong Law of Large Numbers. **[2 marks]**
   (b) Let $X_1, X_2, \cdots$ be i.i.d. ± 1 valued random variables with $P(X_i = 1) = p > \frac{1}{2}$. Consider the partial sums $S_k = X_1 + \cdots + X_k,\ \ k \geq 1$. Let $T_n = \inf\{k : S_k = n\}$ be the first time that the random walk hits n. Show that **[5 marks]**
$$\frac{T_n}{n} \xrightarrow{a.s.} \frac{1}{2p - 1} \text{ as } n \to \infty.$$

7. (a) State the Central Limit Theorem. **[2 marks]**
   (b) Let $X_1, X_2, \cdots, X_{100}$ be i.i.d. random variables with $\mathbf{P}(X_1 = 1) = 0.6$, $\mathbf{P}(X_1 = -1) = 0.4$. Give an approximate value of the probability $P(X_1 + X_2 + \cdots + X_{100} > 10)$ in terms of $\Phi(x)$ (the CDF of the $N(0, 1)$ distribution) using the Central Limit Theorem. **[4 marks]**

8. {**Note:** In the following question, you might use the following fact without proof: Let $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ be functions and X, Y, Z, W be mutually independent random variables. Then the random variables $f(X, Y)$ and $g(Z, W)$ are independent. }

   Let X_1, X_2, Y_1, Y_2 be mutually *independent* random variables defined on a probability space. Assume that X_1 has the same distribution as Y_1, and X_2 has the same distribution as Y_2. Let $f : \mathbf{R}^2 \to \mathbf{R}$, and let $Z = f(X_1, X_2)$ be a random variable such that $E(Z^2) < \infty$.
   (a) Show that **[1 mark]**
$$\text{Var}(Z) = \mathbf{E}\left[f(X_1, X_2) \cdot \Big(f(X_1, X_2) - f(Y_1, X_2)\Big)\right] + \mathbf{E}\left[f(X_1, X_2) \cdot \Big(f(Y_1, X_2) - f(Y_1, Y_2)\Big)\right].$$

(b) Show that **[2 marks]**

$$\mathbf{E}\left[f(X_1,X_2)\cdot\Big(f(X_1,X_2)-f(Y_1,X_2)\Big)\right]=\frac{1}{2}\mathbf{E}\left[\Big(f(X_1,X_2)-f(Y_1,X_2)\Big)^2\right]. \qquad (\#)$$

(c) Show that **[2 marks]**

$$\mathbf{E}\left[f(X_1,X_2)\cdot\Big(f(Y_1,X_2)-f(Y_1,Y_2)\Big)\right]=-\mathbf{E}\left[f(X_1,Y_2)\Big(f(Y_1,X_2)-f(Y_1,Y_2)\Big)\right].$$

(d) Conclude that **[3 marks]**

$$\mathbf{E}\left[f(X_1,X_2)]\cdot\Big(f(Y_1,X_2)-f(Y_1,Y_2)\Big)\right]\leq\frac{1}{2}\mathbf{E}\left[\Big(f(X_1,X_2)-f(X_1,Y_2)\Big)^2\right],$$

(HINT: Use above part and Cauchy-Schwarz), and therefore

$$\mathrm{Var}(Z)\leq\frac{1}{2}\mathbf{E}\left[\Big(f(X_1,X_2)-f(Y_1,X_2)\Big)^2\right]+\frac{1}{2}\mathbf{E}\left[\Big(f(X_1,X_2)-f(X_1,Y_2)\Big)^2\right].$$

(e) Use (#) with the function $f(x_1,x_2)=x_1$ to conclude that **[2 marks]**

$$\mathrm{Var}(X_1)=\frac{1}{2}\mathbf{E}[(X_1-Y_1)^2].$$